Evolution of magnetic field curvature in the Kulsrud-Anderson dynamo theory

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ABSTRACT

We find that in the kinematic limit the ensemble averaged square of the curvature of magnetic field lines is exponentially amplified in time by the turbulent motions in a highly conductive plasma. At the same time, the ensemble averaged curvature vector exponentially decays to zero. Thus, independently of the initial conditions, the fluctuation field becomes very curved, and the curvature vector becomes highly isotropic.

Subject headings: ISM: magnetic fields — MHD — turbulence — methods: analytical

It was shown by Kulsrud and Anderson (1992) that MHD turbulent dynamo action builds magnetic field energy primarily on scales smaller than the smallest turbulent eddy size (which is the viscosity scale), but still larger than the resistive scale (provided the magnetic Prandtl number is large). This result was found under assumption that the "kinematic" approximation is valid, i.e. the field is weak enough that it does not affect the turbulent motions. In this paper we calculate the evolution of magnetic field curvature within the framework of the Kulsrud-Anderson kinematic dynamo theory. Such calculations are of great interest because a rapid built up of curvature may quickly break down the kinematic approximation and change the dynamo action on very small scales considerably.

Following Kulsrud and Anderson (1992) we make the following assumptions. We use the "kinematic" approximation. We neglect resistivity (infinite magnetic Prandtl number limit). We assume that the turbulence is incompressible, homogeneous, isotropic and static, and we use zero correlation time approximation for the turbulent motions:

$$V_\alpha(t,\mathbf{r}) = \frac{1}{(2\pi)^3}\int V_\alpha(t,\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}, \tag{1}$$

$$\langle V_\alpha(t,\mathbf{k})\rangle = 0, \tag{2}$$

$$\begin{aligned}\langle V^*_\alpha(t',\mathbf{k}')V_\beta(t,\mathbf{k})\rangle &= [J_k(\delta_{\alpha\beta} - \hat{k}_\alpha\hat{k}_\beta) \\ &\quad + i\bar{J}_k\varepsilon_{\alpha\gamma\beta}k_\gamma]\delta(\mathbf{k}'-\mathbf{k})\delta(t'-t).\end{aligned} \tag{3}$$

Here and below $\langle ...\rangle$ means ensemble average, $\delta_{\alpha\beta}$ is the Kronecker symbol, $\varepsilon_{\alpha\beta\gamma}$ is the unit anti-symmetric tensor, $\delta(t'-t)$ and $\delta(\mathbf{k}'-\mathbf{k})$ are the Dirac δ-functions, $\hat{\mathbf{k}} = \mathbf{k}/k$ is a unit vector, and we always assume summation over repeated indices. Functions J_k and $\bar{J}_k$ are the normal and the helical parts of the turbulence, they depend only on the absolute value of $\mathbf{k}$. We make no assumptions about the statistics of initial magnetic field.

Using equations (1) and (3), it is straightforward to calculate the correlation tensors between velocities and their spatial derivatives, taken at the same point of space $\mathbf{r}$ but at different time, t and t':

$$\begin{aligned}
&\langle V_\alpha(t',\mathbf{r})V_\beta(t,\mathbf{r})\rangle = (\eta_T/2\pi)\delta_{\alpha\beta}\delta(t'-t),\\
&\langle V_\alpha V_{\beta;\gamma}\rangle = \alpha\varepsilon_{\alpha\beta\gamma}\delta(t'-t),\\
&\langle V_{\alpha;\beta}V_{\gamma;\delta}\rangle = (\gamma/5)(5\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta\gamma\delta})\delta(t'-t),\\
&\langle V_\alpha V_{\beta;\gamma\delta}\rangle = -\left\langle V_{\alpha;\delta}V_{\beta;\gamma}\right\rangle,\\
&\langle V_{\alpha;\beta}V_{\gamma;\delta\tau}\rangle = \omega\varepsilon_{\alpha\gamma\eta}\delta_{\eta\beta\delta\tau}\delta(t'-t),\\
&\langle V_\alpha V_{\beta;\gamma\delta\tau}\rangle = -\left\langle V_{\alpha;\tau}V_{\beta;\gamma\delta}\right\rangle,\\
&\langle V_{\alpha;\beta\gamma}V_{\delta;\tau\eta}\rangle = (\lambda/15)(7\delta_{\alpha\delta}\delta_{\beta\gamma\tau\eta}\\
&\qquad\qquad - \delta_{\alpha\beta\gamma\delta\tau\eta})\delta(t'-t).
\end{aligned} \tag{4}$$

Here and below, in order to shorten notations, spa-

tial derivatives are assumed to be taken with respect to all indices that are listed after ";" signs, e.g. $V_{\alpha;\beta} = \partial V_\alpha/\partial x_\beta$, $V_{\alpha;\beta\gamma} = \partial^2 V_\alpha/\partial x_\beta \partial x_\gamma$. We have also introduced the symmetric tensors

$$\begin{aligned}\delta_{\alpha\beta\gamma\delta} &= \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma},\\ \delta_{\alpha\beta\gamma\delta\tau\eta} &= \delta_{\alpha\beta}\delta_{\gamma\delta\tau\eta} + \delta_{\alpha\gamma}\delta_{\beta\delta\tau\eta} + \delta_{\alpha\delta}\delta_{\beta\gamma\tau\eta}\\ &\quad + \delta_{\alpha\tau}\delta_{\beta\gamma\delta\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma\delta\tau},\end{aligned} \tag{5}$$

and the constants (Kulsrud & Anderson 1992)

$$\begin{aligned}\eta_T &= \frac{4\pi}{3}\frac{1}{(2\pi)^6}\int J_k\, 4\pi k^2 dk,\\ \alpha &= \frac{1}{3}\frac{1}{(2\pi)^6}\int k^2 \bar{J}_k\, 4\pi k^2 dk,\\ \gamma &= \frac{1}{3}\frac{1}{(2\pi)^6}\int k^2 J_k\, 4\pi k^2 dk,\\ \omega &= \frac{1}{15}\frac{1}{(2\pi)^6}\int k^4 \bar{J}_k\, 4\pi k^2 dk,\\ \lambda &= \frac{1}{7}\frac{1}{(2\pi)^6}\int k^4 J_k\, 4\pi k^2 dk.\end{aligned} \tag{6}$$

The equation for the evolution of magnetic field $\mathbf{B}$ in an incompressible highly conductive fluid (Landau & Lifshitz 1983) is

$$\partial B_\alpha/\partial t = V_{\alpha;\beta}B_\beta - V_\beta B_{\alpha;\beta}.$$

We use this formula to derive the equation for the evolution of the unit vector $\mathbf{b} = \mathbf{B}/B$ tangential to magnetic field lines:

$$\partial b_\alpha/\partial t = V_{\alpha;\beta}b_\beta - V_{\beta;\gamma}b_\alpha b_\beta b_\gamma - V_\beta b_{\alpha;\beta}. \tag{7}$$

Now, for a given point of space we assume $\mathbf{b}$ is known at zero time, $t = 0$, and we solve this equation by iterating it twice in time, similar to the calculations of Kulsrud and Anderson (1992). Considering $t > 0$ as an expansion parameter, we have

$$\mathbf{b}(t) = {}^0\mathbf{b} + {}^1\mathbf{b}(t) + {}^2\mathbf{b}(t), \tag{8}$$

where ${}^0\mathbf{b}$ is the value of $\mathbf{b}$ at zero time, ${}^0\mathbf{b} = \mathbf{b}(0)$, and ${}^1\mathbf{b}(t) \propto t$ and ${}^2\mathbf{b}(t) \propto t^2$ are the first and the second order corrections to $\mathbf{b}(t)$, obtained by iterating equation (7) twice in time,

$$\begin{aligned}{}^1b_\alpha(t) = \int_0^t \big\{ &V_{\alpha;\beta}(t'')\,{}^0b_\beta - V_{\beta;\gamma}(t'')\,{}^0b_\alpha {}^0b_\beta {}^0b_\gamma\\ &- V_\beta(t'')\,{}^0b_{\alpha;\beta}\big\}\, dt'',\end{aligned} \tag{9}$$

$$\begin{aligned}{}^2b_\alpha(t) = \int_0^t &\big\{ V_{\alpha;\beta}(t')\,{}^1b_\beta(t') - V_{\beta;\gamma}(t') \times\\ &\times \big[{}^1b_\alpha(t')\,{}^0b_\beta {}^0b_\gamma + {}^0b_\alpha {}^1b_\beta(t')\,{}^0b_\gamma\\ &+ {}^0b_\alpha {}^0b_\beta {}^1b_\gamma(t')\big] - V_\beta(t')\,{}^1b_{\alpha;\beta}(t')\big\}\, dt'.\end{aligned} \tag{10}$$

The *ensemble averaged* curvature $\boldsymbol{\mathcal{K}} \equiv \big\langle(\mathbf{b}\nabla)\mathbf{b}\big\rangle$, up to second order terms, is

$$\begin{aligned}\mathcal{K}_\alpha(t) &= \big\langle b_\beta(t) b_{\alpha;\beta}(t)\big\rangle = \big\langle {}^0b_\beta {}^0b_{\alpha;\beta}\big\rangle\\ &+ \big[\big\langle {}^2b_\beta\big\rangle {}^0b_{\alpha;\beta} + {}^0b_\beta \big\langle {}^2b_\alpha\big\rangle_{;\beta} + \big\langle {}^1b_\beta {}^1b_{\alpha;\beta}\big\rangle\big]\\ &= {}^0\mathcal{K}_\alpha + t\,\big\{-(7\gamma/5)\,{}^0\mathcal{K}_\alpha - (4\gamma/5)\big\langle {}^0b_\alpha {}^0b_{\beta;\beta}\big\rangle\\ &+ \alpha\varepsilon_{\alpha\beta\gamma}\,{}^0\mathcal{K}_{\gamma;\beta} + (\eta_T/4\pi)\,{}^0\mathcal{K}_{\alpha;\beta\beta}\big\}.\end{aligned} \tag{11}$$

Here, we use expansion (8) to find $\boldsymbol{\mathcal{K}}(t)$. The zero order term is ${}^0\mathcal{K}_\alpha = \big\langle {}^0b_\beta {}^0b_{\alpha;\beta}\big\rangle$, all first order terms average out according to formula (2), and the second order terms are given in brackets [...] on the second line of equation (11).[1] The final result for $\mathcal{K}_\alpha(t)$ was obtained by making use of equations (4), (9) and (10).

Introducing another ensemble averaged vector $\boldsymbol{\mathcal{G}} \equiv \big\langle \mathbf{b}\,\mathrm{div}\,\mathbf{b}\big\rangle$, we write the differential equations for time evolution of $\boldsymbol{\mathcal{K}}(t)$ and $\boldsymbol{\mathcal{G}}(t)$ as

$$\frac{\partial\boldsymbol{\mathcal{K}}}{\partial t} = -\frac{7\gamma}{5}\boldsymbol{\mathcal{K}} - \frac{4\gamma}{5}\boldsymbol{\mathcal{G}} + \alpha(\nabla\times\boldsymbol{\mathcal{K}}) + \frac{\eta_T}{4\pi}\triangle\boldsymbol{\mathcal{K}}, \tag{12}$$

$$\frac{\partial\boldsymbol{\mathcal{G}}}{\partial t} = -\gamma\boldsymbol{\mathcal{K}} - \frac{8\gamma}{5}\boldsymbol{\mathcal{G}} + \alpha(\nabla\times\boldsymbol{\mathcal{G}}) + \frac{\eta_T}{4\pi}\triangle\boldsymbol{\mathcal{G}}. \tag{13}$$

Here, the first equation directly follows from formula (11), and the second equation for the evolution of $\boldsymbol{\mathcal{G}}$ can be found by calculations similar to the calculations that led to equation (11), i.e. by expanding $\boldsymbol{\mathcal{G}}(t)$ up to second order terms, by making use of equations (9), (10) and of equations (4) to carry out ensemble averaging.

In a similar way, and after considerable algebra, we find the differential equations describing the evolution of the ensemble averaged square of the curvature, $\mathcal{K}^2 \equiv \big\langle[(\mathbf{b}\nabla)\mathbf{b}]^2\big\rangle \neq \boldsymbol{\mathcal{K}}^2$,

$$\frac{\partial\mathcal{K}^2}{\partial t} = \frac{16\gamma}{5}\mathcal{K}^2 + \frac{8\gamma}{5}D_\mathcal{K} + \frac{\eta_T}{4\pi}\triangle\mathcal{K}^2 + \frac{12\lambda}{5}, \tag{14}$$

$$\frac{\partial D_\mathcal{K}}{\partial t} = -\frac{7\gamma}{5}D_\mathcal{K} - \frac{4\gamma}{5}D_\mathcal{G} + \frac{\eta_T}{4\pi}\triangle D_\mathcal{K}, \tag{15}$$

$$\frac{\partial D_\mathcal{G}}{\partial t} = -\gamma D_\mathcal{K} - \frac{8\gamma}{5}D_\mathcal{G} + \frac{\eta_T}{4\pi}\triangle D_\mathcal{G}. \tag{16}$$

Here, the differential equations for ensemble averaged scalars $D_\mathcal{K} \equiv \big\langle \mathrm{div}\,[(\mathbf{b}\nabla)\mathbf{b}]\big\rangle = \mathrm{div}\,\boldsymbol{\mathcal{K}}$ and

[1] Note that we use $\langle {}^2b_{\alpha;\beta}\rangle = \langle {}^2b_\alpha\rangle_{;\beta}$, which follows from important identity $\langle (V_{\alpha;\beta_1\ldots\beta_n} V_{\gamma;\delta_1\ldots\delta_m})_{;\tau_1\ldots\tau_p}\rangle \equiv 0$. This reflects $\delta(\mathbf{k}' - \mathbf{k})$ correlation property of the turbulence, see eq. (3). Thus, ensemble averaging and taking derivatives can be exchanged for any second order quantity.

$D_{\mathcal{G}} \equiv \big\langle \mathrm{div}\,[\mathbf{b}\,\mathrm{div}\,\mathbf{b}]\big\rangle = \mathrm{div}\,\boldsymbol{\mathcal{G}}$ can be found by taking the divergence of equations (12) and (13) (see the first footnote on page 2).

We can drop $\alpha(\nabla\times\boldsymbol{\mathcal{K}})$ and $\alpha(\nabla\times\boldsymbol{\mathcal{G}})$ terms in equations (12) and (13) because the helical part of the turbulence, $\bar{J}_k$, is usually negligible on scales of the smallest turbulent eddy.[2] In this case equations (12)–(16) have the following solutions:

$$\begin{bmatrix}\boldsymbol{\mathcal{K}}\\ \boldsymbol{\mathcal{G}}\end{bmatrix} = \mathbf{Q}_1\begin{bmatrix}1\\ -1\end{bmatrix}e^{-3\gamma t/5} + \mathbf{Q}_2\begin{bmatrix}4\\ 5\end{bmatrix}e^{-12\gamma t/5},$$

$$\begin{bmatrix}\mathcal{K}^2\\ D_{\mathcal{K}}\\ D_{\mathcal{G}}\end{bmatrix} = Q_3\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}e^{16\gamma t/5} + Q_4\begin{bmatrix}8\\ -19\\ 19\end{bmatrix}e^{-3\gamma t/5}$$
$$+\,Q_5\begin{bmatrix}-8\\ 28\\ 35\end{bmatrix}e^{-12\gamma t/5} + \frac{3\lambda}{4\gamma}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\big(e^{16\gamma t/5}-1\big).$$

Here, the functions $Q_i(t,\mathbf{r})$, $i=1,2,3,4,5$, are the solutions of the same simple diffusion equation

$$\frac{\partial Q_i}{\partial t} = \frac{\eta_T}{4\pi}\triangle Q_i(t,\mathbf{r}), \quad i=1,2,3,4,5 \qquad (17)$$

with different initial conditions:

$$\mathbf{Q}_1(0,\mathbf{r}) = (1/9)\big[5\boldsymbol{\mathcal{K}} - 4\boldsymbol{\mathcal{G}}\big]_{t=0},$$
$$\mathbf{Q}_2(0,\mathbf{r}) = (1/9)\big[\boldsymbol{\mathcal{K}} + \boldsymbol{\mathcal{G}}\big]_{t=0}.$$
$$Q_3(0,\mathbf{r}) = \big[\mathcal{K}^2 + (1/133)(48D_{\mathcal{K}} - 8D_{\mathcal{G}})\big]_{t=0},$$
$$Q_4(0,\mathbf{r}) = (1/171)\big[-5D_{\mathcal{K}} + 4D_{\mathcal{G}}\big]_{t=0},$$
$$Q_5(0,\mathbf{r}) = (1/63)\big[D_{\mathcal{K}} + D_{\mathcal{G}}\big]_{t=0},$$

We see that the ensemble averaged square of the curvature, $\mathcal{K}^2$, exponentially grows with rate $16\gamma/3$. This is even faster than the rate of magnetic field energy growth, 2γ (Kulsrud & Anderson 1992). At the same time, the ensemble averaged curvature vector, $\boldsymbol{\mathcal{K}}$, exponentially decays with rate $-3\gamma/5$. Therefore, $\boldsymbol{\mathcal{K}}$ becomes highly isotropic. According to diffusion equation (17), we find that $\mathcal{K}^2$ and $\boldsymbol{\mathcal{K}}$ become homogeneous on scales L after a diffusion time $\sim 4\pi L^2/\eta_T$.

Let consider the case when the initial magnetic field is constant in space, $\mathbf{B} = \mathrm{const}$ and $\mathbf{b} = \mathrm{const}$. Then $Q_i(t,\mathbf{r}) = 0$, $i=1,2,3,4,5$, and we find that $\mathcal{K}^2 = (3\lambda/4\gamma)\big(e^{16\gamma t/5}-1\big)$. Therefore, even if there is no initial curvature, the field quickly becomes very curved. The curvature first develops linearly in time, $\mathcal{K}^2 \approx 12\lambda t/5$, because of second order spatial derivatives of the turbulent velocities, represented by the "battery" term $12\lambda/5$ in equation (14). Second, at time $t\sim 5/16\gamma$ the averaged curvature reaches the smallest turbulent eddy scale size, $l_{\rm eddy}\sim\gamma/\lambda$, and afterwards exponentiates rapidly.

It was recently suggested by Steven Cowley, on the basis of numerical simulations, that on scales smaller than the smallest turbulent eddy the turbulent motions can only stretch magnetic field lines. It was argued that the rapid exponential increase of the fluctuation field wavenumber $k = (k_\perp^2 + k_\parallel^2)^{1/2}$ found by Kulsrud and Anderson (1992) is mainly due to the increase of $k_\perp$, the wave number perpendicular to the magnetic field lines, while the parallel wave number $k_\parallel$ stays approximately equal to the smallest eddy size, so that $k_\perp \gg k_\parallel$. The results we obtained, that the ensemble (volume) averaged curvature squared, $\mathcal{K}^2$, increases exponentially in time, could be consistent with the folding nature of small-scale fields. For example, the curvature could be comparable to the smallest turbulent eddy size scale over the bulk of the volume, and be enormously large over the remaining small region of the volume (Schekochihin & Cowley 2001).

The consequence of the rapid development of curvature on size scales smaller than the smallest turbulent eddy size scale is that the Lorentz force and viscosity stress on these very small scales become important before energy equipartition, between the field and the turbulence, occurs on the eddy scale. Thus, the magnetic energy at these very small scales may be suppressed at least to some extent.

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[2]E.g. in a galaxy, if the turbulence is Kolmogoroff, $\bar{J}_k \propto k^{-7}$ and $J_k \propto k^{-13/3}$ (Kulsrud & Anderson 1992), these smallest turbulent scales make the largest contributions to α and γ, see eqs. (6).

REFERENCES

Landau, L. D., & Lifshitz, E. M., 1983, *Electrodynamics of continuous media*, Oxford; New York: Pergamon

Kulsrud, R. M., & Anderson, S. W., 1992, ApJ, **396**, 606

Schekochihin, A. A., & Cowley, S. C., 2001, private communication